Discrete Mathematics 128 (1994) 165–172
North-Holland

The square of a chordal graph*

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Received 11 December 1990
Revised 27 March 1992

Abstract

We introduce the closed-neighborhood intersection multigraph as a useful multigraph version of the square of a graph. We characterize those multigraphs which are squares of chordal graphs and include an algorithm to go from the squared chordal graph back to its (unique!) square root. This becomes particularly simple in the case of k-trees, with the case of trees evoking a 1960 paper by Harary and Ross (1960) titled 'The Square of a Tree'.

1. Introduction

In 1960, Harary and Ross [8] characterized graphs which are the squares of trees and gave an algorithm for working from a squared tree back to its (unique!) square root. The characterization is somewhat bulky and involves finding a one-to-one correspondence between the nonedge cliques and the nonsimplicial vertices of the given graph. A shorter matrix version was given by Mukhopadhyay [10] in 1976, still requiring finding the same sort of one-to-one correspondence. We now introduce a multigraph version – the closed-neighborhood multigraph $N[G]$ – of the square of a graph G which subsumes the essence of these characterizations of the square of a tree and for which the algorithm to find the square root tree becomes truly trivial: The edges of T are precisely the multiplicity-2 edges of $N[G]$. The tree result is an instant corollary of a result for k-trees, which is in turn a special case of our general result for chordal graphs.

We basically follow the standard terminology of the texts by Harary [5] and Golumbic [3]. Given any graph G, its *square graph* G^2 (first defined in [8]) has the

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* Research partially supported by the Office of Naval Research contracts N00014-90-J-1860 (for FH) and N00014-88-K-0163 (for TAM).

SSDI 0012-365X(92)00453-R

same vertex set G, with two vertices adjacent in G^2 whenever they are at distance 1 or 2 in G. The name 'square' is motivated as follows: If $A=A(G)$ and $A(G^2)$ are the adjacency matrices of G and G^2, then $A(G^2)$ can be obtained from the matrix $M=A^2+A$ by replacing each diagonal entry of M by 0 and then each remaining positive entry by 1.

An alternative approach is more closely tied to matrix squaring: Suppose $P(G)=P$ is the pseudograph formed by placing a loop at each vertex of G, so P is the reflexive closure of the symmetric relation G and $A(P)=A(G)+I$. Then each entry a_{ij} of $[A(P)]^2$ records the number of length-2 walks from v_i to v_j, and $A(P^2)$ can be obtained from $[A(P)]^2$ by replacing each positive entry by 1. This is the approach used in [4] and developed in detail in the book [6].

Our object is to develop a more powerful multigraph version. As in [5], let $N[v]$ and $N(v)$ be the *closed neighborhood* and *open neighborhood*, respectively, of vertex v of G. We define the *closed-neighborhood multigraph* $N[G]$ to have vertex set $\{N[v]: v\in V(G)\}$, with vertices $N[u]$ and $N[v]$ joined by a multiple edge (called a *multiedge*) consisting of $|N[u]\cap N[v]|$ parallel edges. By identifying each vertex $N[v]$ of $N[G]$ with the vertex v of G, the underlying graph of $N[G]$ is seen to be G^2. We consider $N[G]$ to be the multigraph version of G^2. Neighborhood multigraphs are characterized in [9].

Observe that if we had defined $N[G]$ as a pseudograph with $|N[v_i]|$ parallel loops at the vertex $N[v_i]$ and if the entry a_{ij} of the adjacency matrix $A(N[G])$ recorded the multiplicity of the multiedge joining v_i with v_j in $N[G]$, then $A(N[G])=[A([P(G)])]^2$ with the actual integer entries meaningful.

In Section 2, we characterize those multigraphs M for which there is a chordal graph G such that $M=N[G]$, and develop an algorithm for producing the unique graph G from M. (Observe that, even using multigraphs, a nonchordal graph G need not be uniquely determined by $N[G]$, e.g. $N[K_{3,3}]\cong N[\overline{C_6}]$.) We see in Section 3 how this becomes simpler when G is a k-tree, and we specify the connection with the Harary and Ross [8] characterization of the square of a tree.

2. Closed-neighborhood multigraphs of chordal graphs

For distinct vertices u and v of multigraph M, let $\mu(u, v)$ be the multiplicity of the multiedge uv, or zero if u and v are not adjacent. For any vertex v of M, define the *strength of* v to be $s(v)=\max\{\mu(u,v): u\in V(M)\}$, and call a multiedge uv a *strong* v*-multiedge* whenever $\mu(u,v)=s(v)$. Define the *strength of* M to be $s(M)=\max\{s(v): v\in V(M)\}$. A vertex v of multigraph M is called *simple* if the number of strong v-multiedges is $s(v)-1$ and the endpoints of these strong v-multiedges induce a complete subgraph in the underlying graph $G(M)$. For instance, each of v_1, v_2, v_3 and v_6 is simple in the strength-4 multigraph M of Fig. 1.

If v is simple in M, let $M\backslash v$ be obtained from M by deleting v and then lowering by one the multiplicity of each multiedge ww' such that vw and vw' are strong v-

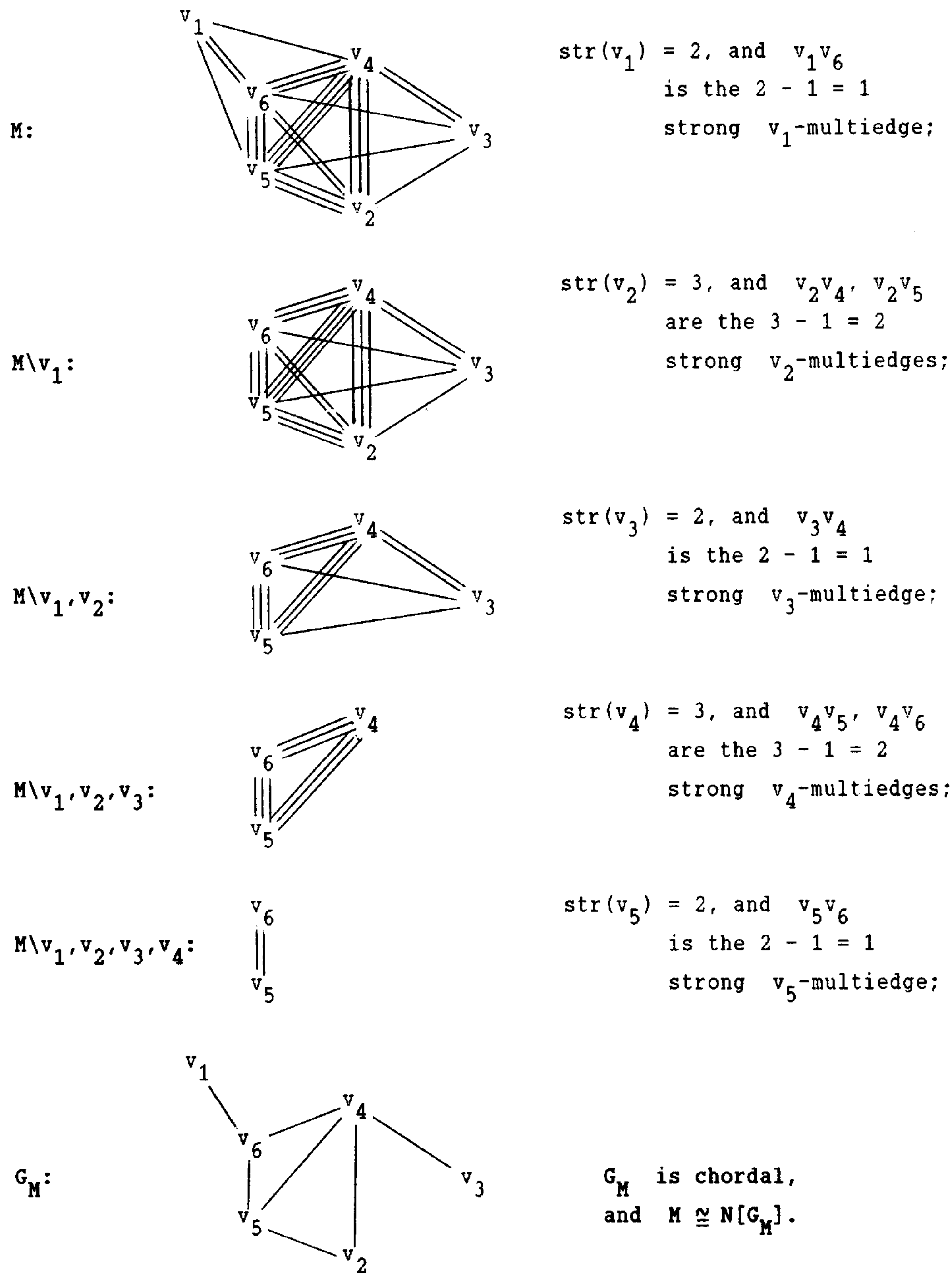
M:
str(v1) = 2, and v1v6 is the 2 - 1 = 1 strong v1-multiedge;
M\v1:
str(v2) = 3, and v2v4, v2v5 are the 3 - 1 = 2 strong v2-multiedges;
M\v1,v2:
str(v3) = 2, and v3v4 is the 2 - 1 = 1 strong v3-multiedge;
M\v1,v2,v3:
str(v4) = 3, and v4v5, v4v6 are the 3 - 1 = 2 strong v4-multiedges;
M\v1,v2,v3,v4:
str(v5) = 2, and v5v6 is the 2 - 1 = 1 strong v5-multiedge;
GM:
GM is chordal, and M ≅ N[GM].
v1
v2
v3
v4
v5
v6

Fig. 1.

multiedges, and define $M\backslash v_1, \dots, v_{i+1}$ recursively as $(M\backslash v_1, \dots, v_i)\backslash v_{i+1}$. For $v \in V(G)$, let $G-v$ be obtained from G by deleting v, and define $G-v_1, \dots, v_{i+1}$ recursively as $(G-v_1, \dots, v_i)-v_{i+1}$. The following lemma shows how simple vertices of $N[G]$ play a role similar to the simplicial vertices of G, where a vertex v is *simplicial* in G if $N(v)$ is a complete subgraph of G.

Lemma 2.1. *If vertex v is simplicial in a graph G, then vertex $N[v]$ is simple in $N[G]$, and $N[G]\backslash N[v] \cong N[G-v]$, and vw is an edge of G if and only if $N[v]N[w]$ is a strong $N[v]$-multiedge of $N[G]$.*

Proof. First let v be simplicial in G. Let $v_1, \dots, v_r$ be the neighbors of v and $u_1, \dots, u_s$ be the vertices at distance 2 from v in G. So each neighbor of the vertex $N[v]$ of $N[G]$ will be of the form $N[v_i]$ or $N[u_i]$. Since v is simplicial, $N[v] \cap N[v_i] = \{v, v_1, \dots, v_r\}$, each $\mu(N[v], N[v_i]) = r+1$, and there are exactly r such multiedges $N[v]N[v_i]$ in $N[G]$. As $N[v] \cap N[u_i] \subset \{v_1, \dots, v_r\}$, every other multiedge incident with $N[v]$ in $N[G]$ will have multiplicity less than $r+1$. Thus, $s(N[v]) = r+1$ and there are exactly r strong $N[v]$-multiedges. If $i \neq j$, $\mu(N[v_i], N[v_j]) = |N[v_i] \cap N[v_j]| \geqslant r+1$, so $N[v_i]$ and $N[v_j]$ are adjacent, and hence $N[v]$ is simple in $N[G]$.

The remainder of the proof is omitted as it is straightforward to check. □

Call an ordering $v_1, \dots, v_n$ of the vertices of M a *simple elimination ordering* whenever, for each $i<n$, v_i is simple in $M\backslash v_1, \dots, v_{i-1}$. For instance, the ordering $v_1, \dots, v_n$ in Fig. 1 is a simple elimination ordering for M. When such a simple elimination ordering exists for M, let G_M be the graph on the same vertex set as M, with an edge v_iv_j for $i<j$ whenever v_iv_j is one of the strong v_i-multiedges in $M\backslash v_1, \dots, v_{i-1}$.

Theorem 2.2. *A multigraph M is the closed-neighborhood multigraph of a chordal graph if and only if M has a simple elimination ordering, the resulting G_M is chordal, and $M \cong N[G_M]$. Moreover, when this happens, G_M is the unique graph G such that $M \cong N[G]$.*

Proof. Suppose $M = N[G]$, where G is chordal. By a theorem of Fulkerson and Gross [2] (= [3, Theorem 4.1]), G being chordal is equivalent to the existence of a 'simplicial elimination ordering' $v_1, \dots, v_n$ of the vertices of G such that each v_i is simplicial in $G-v_1, \dots, v_{i-1}$. We argue by induction on n, the order of G, with the $n=1$ case being trivial. Since the subgraph $G' = G - v_1$ is chordal, Lemma 2.1 implies that $N[v_1]$ is simple in M, so we can put $M' = M\backslash N[v_1]$ and we see that $M' \cong N[G']$. The inductive hypothesis implies that $G_{M'}$, is chordal and is the unique graph whose closed-neighborhood multigraph is M'. Thus $G' \cong G_{M'}$ and so, by Lemma 2.1, $G \cong G_M$. □

3. Closed-neighborhood multigraphs of k-trees

The notion of a *k-tree* is defined recursively as follows: The complete graph of order k is the only k-tree of order less than or equal to k, and the k-trees of order $n+1>k$ are obtained from the k-trees of order n by adjoining a new simplicial vertex adjacent precisely to the vertices of some complete subgraph of order k. (Such k-trees were introduced in [7] and are surveyed in [1, 11, 12].) Therefore, 1-trees are precisely the usual trees and each k-tree G has a simplicial elimination order $v_1,\dots,v_n$ in which each v_i having $1\leqslant i\leqslant n-k$ has degree k in $G\backslash v_1,\dots,v_{i-1}$. Therefore, every k-tree is a chordal graph.

If G is a k-tree, then the strong $N[v_i]$-multiedges of $N[G]\backslash N[v_1],\dots,N[v_{i-1}]$ always have strength $k+1$ and so we can go immediately from $N[G]$ back to G. For any multigraph M, let $G_{M,i}$ denote the graph on the same vertex set as M, with two vertices u and v adjacent whenever $\mu(u,v)>i$ in M; see Fig. 2. The following corollaries follow from the proof of Theorem 2.2.

Corollary 3.1. *A multigraph M is the closed-neighborhood multigraph of a k-tree if and only if $G_{M,k}$ is a k-tree and $M\cong N[G_{M,k}]$.*

Corollary 3.2. *A multigraph M is the closed-neighborhood multigraph of a tree if and only if M has strength two and the double edges of M form a tree T such that $M\cong N[T]$.*

An example of a tree T and $N[T]$ is shown in Fig. 3. Corollary 2.2 can be turned into a recognition algorithm for squared trees and for constructing the root graph of a squared tree as follows. (Failure of any 'check' means that the given graph G is not a squared tree.) In step (I) of the algorithm, the (inclusion-maximal!) cliques of G can be efficiently found while testing whether G is chordal; see [3, Algorithms 4.2 and 4.3].

Recognition algorithm for the square of a tree

(I) Check that G is chordal and find all its cliques.

(II) Check that no edge of G is in more than two cliques and double those which are in exactly two.

(III) Check that the doubled edges form a tree which dominates the vertex set of G.

(IV) Check that each vertex not on a doubled edge is simplicial and that its closed neighborhood contains either (i) one leaf edge of the doubled tree or (ii) two or more mutually adjacent doubled edges.

Correctness Proof. Suppose first that G is the square of a tree T. We show that each 'check' in the algorithm will succeed. Since G is the intersection graph of the subtrees (stars) of T consisting of the join $v+N(v)$ for each $v\in V(T)$, G is chordal by [3, Theorem 4.8]. Since G is the intersection graph of the closed neighborhoods of T, the internal vertices of T correspond to the cliques of G, with each internal vertex $v\in V(T)$ corresponding to $\langle N[u]\colon v\in N[u]\rangle$. Similarly, the leaves of T correspond to the

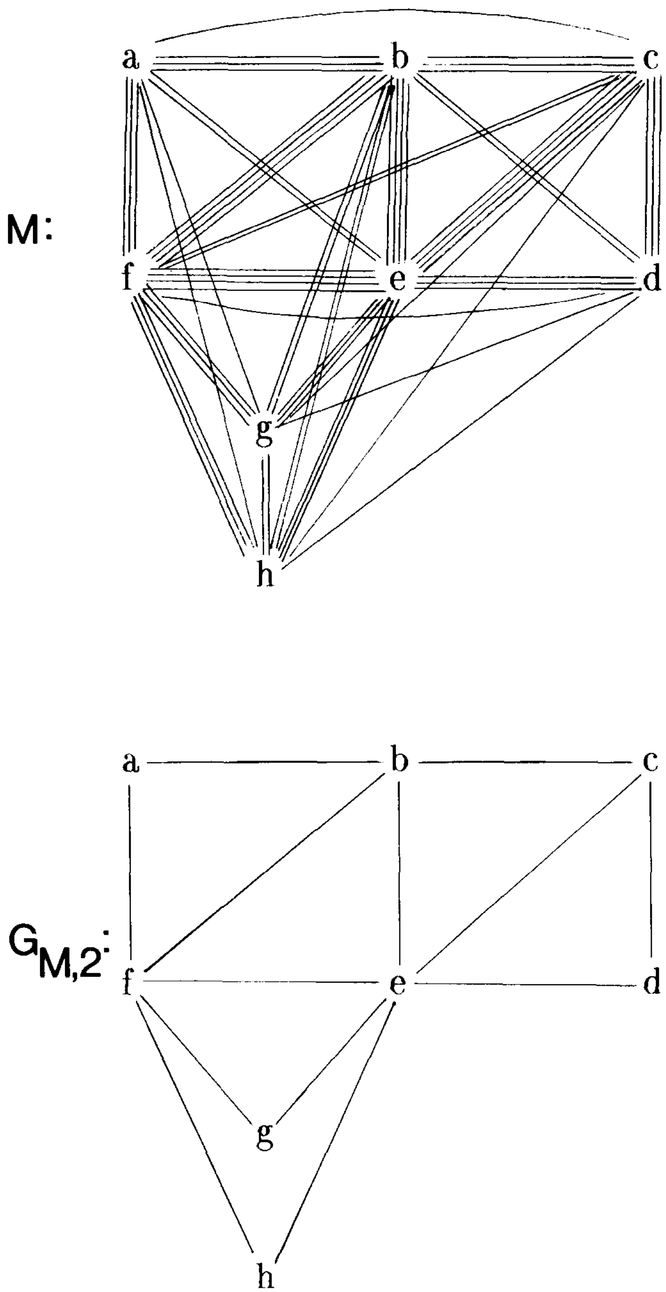

Fig. 2.

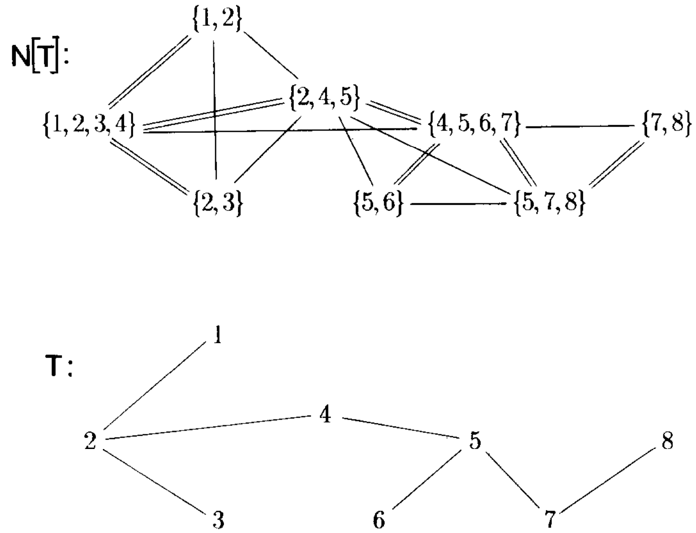

Fig. 3.

simplicial vertices of G, with each leaf $v \in T$ having $N[v]=\{v, w\}$ corresponding to $N[v] \in V(G)$ in the unique clique $\langle N[u]: w \in N[u] \rangle$ in G. No edge of G can be in three cliques, since that would correspond to two closed neighborhoods of T containing three common internal vertices of T, and so to there being a triangle in T. The edges of G which are doubled in step (II) correspond to adjacent internal vertices and so to the internal edges of T. Let T' be the tree of doubled edges in (III). Each vertex of G not in T' is in a unique clique of G and so corresponds to a leaf of T. Such a leaf is as in case (i) of (IV) if its neighbor in T is a leaf of T', and is as in case (ii) otherwise. (In the example of Fig. 3, $\{7, 8\}$ and $\{5, 6\}$ are vertices of types (i) and (ii), respectively.)

Finally, suppose a graph G satisfies each of the checks in the algorithm. The subtree T' of edges which are doubled in (III) can be extended to a spanning tree T of G using (IV) as follows: In case (i), double the edge joining the simplicial vertex to the adjacent leaf vertex of T'; in case (ii), double the edge joining the simplicial vertex with the adjacent internal vertex of T' which is incident with each of the mutually adjacent doubled edges. The multigraph obtained from G by doubling these edges is precisely the closed-neighborhood multigraph M with tree T of doubled edges as in Corollary 3.2. $\square$.

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